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Lorentz-Covariant Hamiltonian Formalism

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The dynamics of a classical system can be expressed by means of Poisson brackets. In this paper we generalize the relation between the usual noncovariant Hamiltonian and the Poisson brackets to a covariant Hamiltonian and new brackets in the frame of Minkowski space. These brackets can be related to those used by Feynman in his derivation Maxwell's equations. The case of curved space is also considered with the introduction of Christoffel symbols, covariant derivatives, and curvature tensors.

1. INTRODUCTION

A remarkable formulation of classical dynamics is provided by Hamiltonian mechanics. This is an old subject. However, new discoveries are still been made; we quote two examples among several: the Arnold duality transformations, which generalize the canonical transformations,[(1,2)] and the extensions of Poisson brackets to differential forms and multivector fields by Cabras and Vinogradov.[(3)] In this context the transition from classical to relativistic mechanics raises the question of Hamiltonian covariance, the physical significance of which is discussed for example by Goldstein.[(4)]

In the first part of this paper we briefly recall the Poisson brackets approach and the covariant Hamiltonian formalism. Then we introduce new brackets to study the dynamics associated to this covariant Hamiltonian, which define an algebraic structure between position and velocity, and does not have an explicit formulation. We examine the close link between these brackets and those used by Feynman for his derivation of the Maxwell equations.[(5–8)] A very interesting way to arrive at the same sort of result was found by Souriau in the frame of his symplectic classical mechanics.[(9)] In

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the final part of this work we consider the dynamics in curved space, using Christoffel symbols, covariant derivatives, and curvature tensors expressed in terms of these brackets.

2. BRIEF REVIEW OF ANALYTIC MECHANICS

2.1. Poisson Brackets

The dynamics of a classical particle in a 3-dimensional flat space with vector position q^i and vector momentum p_i ($i = 1, 2, 3$) is defined by the Hamilton equations

$$\begin{cases} \dot{q}^i = \dfrac{dq^i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q^i} \end{cases} \tag{1}$$

where the Hamiltonian $H(q^i, p_i)$ is a form on the phase space (the cotangent fiber space). They can be also expressed in a symmetric manner by means of Poisson brackets:

$$\begin{cases} \dot{q}^i = \{q^i, H\} \\ \dot{p}_i = \{p_i, H\} \end{cases} \tag{2}$$

These brackets are naturally defined as skew-symmetric bilinear maps on the space of functions on the phase space in the following form:

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \tag{3}$$

2.2. Covariant Hamiltonian

Except in the electromagnetic situation, the Hamiltonian is not the total energy when it is time-dependent, and its generalization to relativistic problems with the M_4 Minkowski space is not trivial because it is not Lorentz covariant.

In the electromagnetic case the answer to this situation is given by the introduction of the following covariant expression[4]:

$$H = u^{\mu} p_{\mu} - L = u^{\mu}\left(m u_{\mu} + \frac{q}{c} A_{\mu}\right) \tag{4}$$

where L is the usual invariant electromagnetic Lagrangian:

$$L = \frac{1}{2} m u^{\mu} u_{\mu} + \frac{q}{c} u^{\mu} A_{\mu} \tag{5}$$

and u_{μ} is the 4-velocity defined by means of the proper time t_p, here used as an invariant parameter:

$$u^{\mu} = \frac{dx^{\mu}}{dt_p} \tag{6}$$

Finally, we have the covariant Hamiltonian

$$H = \frac{1}{2} m u^{\mu} u_{\mu} \tag{7}$$

with the corresponding eight Hamilton equations:

$$\begin{cases} \dfrac{\partial H}{\partial p_{\mu}} = \dfrac{dx^{\mu}}{dt_p} = u^{\mu} \\ \dfrac{\partial H}{\partial x^{\mu}} = -\dfrac{dp_{\mu}}{dt_p} \end{cases} \tag{8}$$

It is interesting to recall that this structure is only possible in the situation where the potential can be put in a covariant manner as in the theory of electromagnetism.

3. LORENTZ COVARIANT HAMILTONIAN AND BRACKETS FORMALISM

Now we want to generalize the relation between the usual noncovariant relativistic Hamiltonian and the Poisson brackets to a covariant Hamiltonian H and new formal brackets introduced in the frame of the Minkowski space. It is important to remark that, in a different manner, Bracken also studied the relation between this Feynman problem and the Poisson brackets.[10]

In this context a "dynamic evolution law" is given by means of a one real-parameter group of diffeomorphic transformations:

$$g\ (IR \times M_4) \to M_4: \quad g(\tau, x) = g^{\tau} x = x(\tau)$$

The "velocity vector" associated to the particle is naturally introduced by

$$\dot{x}^{\mu} = \frac{d}{d\tau} g^{\tau} x^{\mu} \tag{9}$$

where the "time" τ is not identified with the proper time, as we shall see later. The derivative with respect to this parameter of an arbitrary function defined on the tangent bundle space can be written, by means of the covariant Hamiltonian, as

$$\frac{df(x, \dot{x},\tau)}{d\tau} = [H, f(x, \dot{x},\tau)] + \frac{\partial f(x, \dot{x},\tau)}{\partial \tau} \tag{10}$$

where for H we take the following definition:

$$H = \frac{1}{2}\, m \frac{dx^{\mu}}{d\tau} \frac{dx_{\mu}}{d\tau} = \frac{1}{2}\, m\dot{x}^{\mu}\dot{x}_{\mu} \tag{11}$$

Equation (10) giving the dynamics of the system is the definition of our new brackets structure, and is the fundamental equation of this paper.

We require for these new brackets the usual first Leibniz law:

$$[A, BC] = [A, B]C + [A, C]B \tag{12}$$

and the skew symmetry:

$$[A, B] = -\,[B, A] \tag{13}$$

where the quantities A, B, and C depend of x^{μ} and $\dot{x}^{\mu}$.

In the case of the vector position $x^{\mu}(\tau)$ we have from (10)

$$\dot{x}^{\mu} = [H, x^{\mu}] = m[\dot{x}^{\nu}, x^{\mu}]\, \dot{x}_{\nu} \tag{14}$$

and we easily deduce that

$$m[\dot{x}^{\nu}, x^{\mu}] = g^{\mu\nu} \tag{15}$$

where $g^{\mu\nu}$ is the metric tensor of the Minkowski space.

As in the Feynman approach, the time parameter is not the proper time. To see this we borrow Tanimura's argument.[6] Consider the relation

$$g^{\mu\nu} \frac{dx^{\mu}}{dt_p} \frac{dx^{\nu}}{dt_p} = 1 \tag{16}$$

which implies

$$\left[\dot{x}^{\lambda}, g^{\mu\nu} \frac{dx^{\mu}}{dt_p} \frac{dx^{\nu}}{dt_p}\right] = 0 \tag{17}$$

and is in contradiction with

$$\left[\dot{x}^{\lambda}, g^{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}\right] = -\frac{2}{m}\, \dot{x}^{\lambda} \tag{18}$$

But differently from Feynman, the fact that $g^{\mu\nu}$ is the metric is a consequence of the formalism and is not imposed by hand. In addition, contrary to Feynman, we do not need to impose the Leibniz condition:

$$\frac{d}{d\tau}[A, B] = \left[\frac{dA}{d\tau}, B\right] + \left[A, \frac{dB}{d\tau}\right] \tag{19}$$

(A and B being position- and velocity-dependent functions) because the time derivative is given by the fundamental equation (10).

We impose the usual locality property:

$$[x^\mu, x^\nu] = 0 \tag{20}$$

which directly gives for an expandable function of the position or the velocity the following useful relations:

$$\begin{cases} [x^\mu, f(\dot{x})] = -\dfrac{1}{m}\dfrac{\partial f(\dot{x})}{\partial \dot{x}_\mu} \\ [\dot{x}^\mu, f(x)] = \dfrac{1}{m}\dfrac{\partial f(x)}{\partial x_\mu} \end{cases} \tag{21}$$

which reduce in the particular cases of the position and velocity to

$$\begin{cases} [x^\mu, \dot{x}^\nu] = -\dfrac{1}{m} g^{\mu\rho}\dfrac{\partial \dot{x}^\nu}{\partial \dot{x}^\rho} = -\dfrac{1}{m}\dfrac{\partial \dot{x}^\nu}{\partial \dot{x}_\mu} = -\dfrac{g^{\mu\nu}}{m} \\ [\dot{x}^\mu, x^\nu] = \dfrac{1}{m} g^{\mu\rho}\dfrac{\partial x^\nu}{\partial x^\rho} = \dfrac{1}{m}\dfrac{\partial x^\nu}{\partial x_\mu} = \dfrac{g^{\mu\nu}}{m} \end{cases} \tag{22}$$

To compute the brackets between two components of the velocity we require in addition the Jacobi identity:

$$[[\dot{x}^\mu, \dot{x}^\nu], x^\rho] + [[x^\rho, \dot{x}^\mu], \dot{x}^\nu] + [[\dot{x}^\nu, x^\rho], \dot{x}^\mu] = 0 \tag{23}$$

which by using (15) gives

$$[\dot{x}^\mu, \dot{x}^\nu] = -\frac{N^{\mu\nu}(x)}{m} \tag{24}$$

where $N^{\mu\nu}(x)$ is a skew symmetric tensor.

The second derivative of the position vector is

$$\ddot{x}^\mu = \frac{d\dot{x}^\mu}{d\tau} = [H, \dot{x}^\mu] = N^{\mu\nu}\dot{x}_\nu \tag{25}$$

and we write

$$F^{\mu\nu} = \frac{m}{q} N^{\mu\nu} \tag{26}$$

in order to recover the Lorentz equation of motion.

Remark 1. We can easily calculate

$$[H, H] = \frac{1}{4}m^2[\dot{x}_\mu\dot{x}^\mu, \dot{x}_\nu\dot{x}^\nu] = -q\dot{x}_\mu\dot{x}_\nu F^{\mu\nu} = 0 \tag{27}$$

and then deduce

$$\frac{dH}{d\tau} = \frac{\partial H}{\partial \tau} \tag{28}$$

which is the expected result.

In the same manner, we get for the 4-orbital momentum

$$\begin{aligned}\frac{dL^{\mu\nu}}{d\tau} &= m\frac{d}{d\tau}(\dot{x}^\mu\dot{x}^\nu - \dot{x}^\nu\dot{x}^\nu) = m(\dot{x}^\mu\ddot{x}^\nu - \ddot{x}^\mu\dot{x}^\nu)\\ &= q(\dot{x}^\nu F^{\nu\rho}\dot{x}_\rho - \dot{x}^\nu F^{\mu\rho}\dot{x}_\rho) = [H, L^{\mu\nu}]\end{aligned} \tag{29}$$

as expected.

4. MAXWELL EQUATIONS

Our formal construction will give the Maxwell equations because it leads to the fundamental result (15), which is the starting point of Feynman's proof of the first group of Maxwell equations. The difference is that our main property is equation (10) and not the Leibniz rule (19). So our derivation will be obtained differently and will give in addition the two groups of Maxwell equations.

To be general, we choose as in ref. 8 the following definition for the gauge curvature:

$$[\dot{x}^\mu, \dot{x}^\nu] = -\frac{1}{m^2}(qF^{\mu\nu} + g{*}F^{\mu\nu} \tag{30}$$

where g will be interpreted as the magnetic charge of the Dirac monopole, the *-operation being the Hodge duality.

A simple derivative gives

$$\frac{d(qF^{\mu\nu}(x) + g{*}F^{\mu\nu}(x))}{d\tau} = q\partial^\rho F^{\mu\nu}(x)\dot{x}_\rho + g\partial^{\rho *}F^{\mu\nu}(x)\dot{x}_\rho \tag{31}$$

and means of the fundamental relation (10) we obtain

$$\frac{d(qF^{\mu\nu}(x) + g{}^*F^{\mu\nu}(x))}{d\tau} = [H, qF^{\mu\nu}(x) + g{}^*F^{\mu\nu}x)]$$
$$= -\frac{m^3}{q}[\dot{x}^\rho, [\dot{x}^\mu, \dot{x}^\nu]]\dot{x}_\rho \tag{32}$$

Now using the Jacobi identity, we rewrite this expression as

$$\frac{d(qF^{\mu\nu}(x) + g{}^*F^{\mu\nu}(x))}{d\tau} = \frac{m^3}{q}([\dot{x}^\mu, [\dot{x}^\nu, \dot{x}^\rho]]\dot{x}_\rho + [\dot{x}^\nu, [\dot{x}^\rho \dot{x}^\mu]]\dot{x}_\rho)x_\rho$$
$$= -q(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu}x)\dot{x}_\rho$$
$$- g(\partial^{\mu *}F^{\nu\rho} + \partial^{\nu *}F^{\rho\mu}x)\dot{x}_\rho \tag{33}$$

By comparing equations (31) and (33), we deduce the following field equation:

$$q(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu}) + g(\partial^{\mu *}F^{\nu\rho} + \partial^{\nu *}F^{\rho\mu} + \partial^{\rho *}F^{\mu\nu}) = 0 \tag{34}$$

that is,

$$\begin{cases} \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = gN^{\mu\nu\rho} \\ \partial^{\mu *}F^{\nu\rho} + \partial^{\nu *}F^{\rho\mu} + \partial^{\rho *}F^{\mu\nu} = -qN^{\mu\nu\rho} \end{cases} \tag{35}$$

where $N^{\mu\nu\rho}$ is a tensor to be interpreted.

Using the differential forms language defined on the Minkowski space (M_4), we write the preceding equations in a compact form:

$$\begin{cases} dF = gN \\ d{}^*F = -qN \end{cases} \tag{36}$$

where F and ${}^*F \in \wedge^2(M_4)$ and $N \in \wedge^3(M_4)$.

If we put

$$\begin{cases} gN = -{}^*k \\ qN = {}^*j \end{cases} \tag{37}$$

where j and $k \in \wedge^1(M_4)$, we deduce

$$\begin{cases} \delta F = j \\ dF = -{}^*k \end{cases} \tag{38}$$

δ is the usual codifferential

$$\delta: \quad \wedge^k(M_4) \to \wedge^{k-1}(M_4)$$

defined here as

$$\delta = (-)^{k(4-k+1)+1}(*d*)$$

Interpreting the 1-forms j and k as the electric and magnetic four-dimensional current densities, we obtain the two groups of Maxwell equations in the presence of a magnetic monopole. The situation without monopole is obtained by putting the 1-form k equal to zero.

We easily see by means of the Poincaré theorem that

$$\delta^2 F = \delta j = 0 \tag{39}$$

which is nothing else than the current density continuity equation:

$$\partial_\mu j^\mu = m[\dot{x}_\mu, j^\mu] = 0 \tag{40}$$

From the skew property of the brackets, we can choose

$$j^\mu = \rho \dot{x}^\mu \tag{41}$$

ρ is the charge density, whose dynamic evolution is given by

$$\frac{d\rho}{d\tau} = [H, \rho] = m[\dot{x}^\mu, \rho]\dot{x}_\mu = (\partial^\mu \rho)\dot{x}_\mu = \partial^\mu j_\mu = 0 \tag{42}$$

We see that H automatically takes into account the gauge curvature. It plays the role of a Hamiltonian not with the usual Poisson brackets, but with new four-dimensional brackets which can be related to for example, those used by Feynman in his derivation of the Maxwell equations as published by Dyson.[5]

5. APPLICATION TO CURVED SPACE

In this section we extend the previous analysis to the case of a general space time metric $g_{\mu\nu}(x)$.

In this case we define the covariant Hamiltonian from the usual fundamental quadratic form ds^2 in the following manner:

$$H = \frac{1}{2} m \left(\frac{ds}{d\tau}\right)^2 \quad \frac{1}{2} m g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$$

In the same manner as in Section 3, we can prove the relation between the metric tensor and the bracket structure:

$$m[\dot{x}^\nu, x^\mu] = g^{\mu\nu}(x)$$

The law of motion is

$$
\begin{aligned}
\ddot{x}^{\mu} &= [H, \dot{x}^{\mu}] = \frac{1}{2}\, m[g_{\nu\rho}, \dot{x}^{\mu}]\dot{x}^{\nu}\dot{x}^{\rho} + m[\dot{x}^{\nu}, \dot{x}^{\mu}]\dot{x}_{\nu} \\
&= -\frac{1}{2}\, \partial^{\mu} g_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} - N^{\nu\mu}\dot{x}_{\nu} \qquad (43)
\end{aligned}
$$

where we define $N^{\mu\nu}(x, \dot{x})$ as

$$
[\dot{x}^{\mu}, \dot{x}^{\nu}] = -\frac{N^{\mu\nu}(x, \dot{x})}{m} \qquad (44)
$$

Note that this tensor is now velocity-dependent, in contrast to the Minkowski case.

By means of equations (23) and (43), we deduce the relation

$$
\frac{\partial N^{\mu\nu}}{\partial \dot{x}_{\rho}} = \partial^{\nu} g^{\rho\mu} - \partial^{\mu} g^{\rho\nu} \qquad (45)
$$

Then

$$
N^{\mu\nu}(x, \dot{x}) = -(\partial^{\mu} g^{\rho\nu} - \partial^{\nu} g^{\rho\mu})\dot{x}_{\rho} + n^{\mu\nu}(x) \qquad (46)
$$

where the tensor $n^{\mu\nu}(x)$ is only position dependent. If we introduce this equation in (43), we find

$$
\begin{aligned}
\ddot{x}^{\mu} &= -\frac{1}{2}\, \partial^{\mu} g_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} - (\partial^{\mu} g^{\rho\nu} - \partial^{\nu} g^{\rho\mu})\dot{x}_{\nu}\dot{x}_{\rho} + n^{\mu\nu}(x)x_{\nu} \\
&= \frac{1}{2}\, \partial^{\mu} g^{\nu\rho}\dot{x}_{\nu}\dot{x}_{\rho} - \left(\partial^{\mu} g^{\rho\nu} - \frac{1}{2}\, \partial^{\nu} g^{\rho\mu} - \frac{1}{2}\, \partial^{\rho} g^{\nu\mu}\right)\dot{x}_{\nu}\dot{x}_{\rho} + n^{\mu\nu}(x)\dot{x}_{\nu} \\
&= -\Gamma^{\nu\rho\mu}\dot{x}_{\nu}\dot{x}_{\rho} + n^{\mu\nu}(x)\dot{x}_{\nu} \qquad (47)
\end{aligned}
$$

where we have defined the Christoffel symbols by

$$
\begin{aligned}
\Gamma^{\nu\rho\mu} &= \frac{1}{2}\, ([\dot{x}^{\rho}, [\dot{x}^{\nu}, x^{\mu}]] - [\dot{x}^{\nu}, [\dot{x}^{\rho}, x^{\mu}]] - [\dot{x}^{\mu}, [\dot{x}^{\rho}, x^{\nu}]]) \qquad (48) \\
&= \frac{1}{2}\, (\partial^{\rho} g^{\nu\mu} - \partial^{\nu} g^{\rho\mu} - \partial^{\mu} g^{\rho\nu}) \qquad (49)
\end{aligned}
$$

Comparing this with the usual law of motion of a particle in an electromagnetic field, as in the situation of flat space, we can put

$$
F^{\mu\nu}(x) = \frac{m}{q}\, n^{\mu\nu}(x) \qquad (50)
$$

and get the equation of motion of a particle in curved space:

$$m\,\frac{d\dot{x}^{\mu}}{d\tau} = -m\Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} - qF^{\nu\mu}\dot{x}_{\upsilon} \tag{51}$$

so that

$$[H, \dot{x}^{\mu}] = -\Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} - \frac{q}{m}\,F^{\nu\mu}\dot{x}_{\upsilon} \tag{52}$$

Note the difference between the two tensors $N^{\mu\nu}$ and $F_{\mu\nu}$ whose definitions are

$$\begin{cases}[\dot{x}^{\mu}, \dot{x}^{\nu}] = -\dfrac{N^{\mu\nu}}{m} = -g^{\mu\rho}g^{\nu\sigma}\,\dfrac{N_{\rho\sigma}}{m} \\ [\dot{x}_{\mu}, \dot{x}_{\nu}] = -\dfrac{F_{\mu\nu}}{m} = -g_{\mu\rho}g_{\nu\sigma}\,\dfrac{F^{\rho\sigma}}{m}\end{cases} \tag{53}$$

and more generally

$$\begin{cases}[\dot{x}^{\mu}, f(\dot{x}, \tau)] = \dfrac{N^{\mu\nu}}{m}\,\dfrac{\partial f(\dot{x}, \tau)}{\partial \dot{x}^{\nu}} \\ [\dot{x}_{\mu}, f(\dot{x}, \tau)] = \dfrac{F_{\mu\nu}}{m}\,\dfrac{\partial f(\dot{x}, \tau)}{\partial \dot{x}_{\nu}}\end{cases} \tag{54}$$

As in the case of flat Minkowski space, it is not difficult to recover the two groups of Maxwell equations with or without monopoles. In this last case we must take the following definition for the dual field

$$*F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{55}$$

Now we will show that the covariant derivative and the curvature tensor can be naturally introduced with our formalism.

5.1 Covariant Derivative

As in the flat-space case, the equation of motion can be rewritten in the two following ways:

$$m\,\frac{d\dot{x}^{\mu}}{d\tau} = -m\Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} - qF^{\nu\mu}\dot{x}_{\upsilon} \tag{56}$$

and

$$m\,\frac{d\dot{x}^{\mu}}{d\tau} = m\,\frac{\partial \dot{x}^{\mu}}{\partial x^{\nu}}\,\dot{x}^{\nu} \tag{57}$$

We then put

$$\frac{\partial \dot{x}^{\mu}}{\partial x^{\nu}} = -\Gamma^{\mu}_{\nu\rho}\dot{x}^{\rho} + \frac{q}{m} F^{\mu}_{\ \nu} = [H', \dot{x}^{\mu}] \tag{58}$$

From equation (58), a covariant derivative can be defined by means of the brackets. For an arbitrary vector we put

$$m[\dot{x}_{\nu}, V^{\mu}(x)] = \frac{\partial V^{\mu}(x)}{\partial x^{\nu}} \tag{59}$$

We then define as the usual covariant derivative

$$[D_{\nu}, V^{\mu}] = \frac{\partial V^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho}V^{\rho} \tag{60}$$

and for an arbitrary mixed tensor

$$[D_{\nu}, T^{\mu}_{\sigma}] = \frac{\partial T^{\mu}_{\sigma}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho}T^{\rho}_{\sigma} - \Gamma^{\rho}_{\nu\sigma}T^{\mu}_{\rho} \tag{61}$$

For the particular case of the velocity we get

$$[D_{\nu}, \dot{x}^{\mu}] = \frac{\partial \dot{x}^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\rho} = \frac{q}{m} F^{\mu}_{\ \nu} \tag{62}$$

and in addition we recover the standard result

$$[D_{\nu}, g^{\mu\nu}] = 0$$

5.2. Curvature Tensor

From this definition of the covariant derivative we can naturally express a curvature tensor by means of the brackets. Let us compute the following expressions:

$$\begin{aligned} [D^{\mu}, [D^{\nu}, V^{\rho}]] &= [\dot{x}^{\mu}, \partial^{\nu}V^{\rho} + \Gamma^{\nu\rho}_{\sigma}V^{\sigma}] \\ &\quad + \Gamma^{\mu\nu}_{\alpha}(\partial^{\alpha}V^{\rho} + \Gamma^{\alpha\rho}_{\sigma}V^{\sigma}) + \Gamma^{\mu\rho}_{\alpha}(\partial^{\nu}V^{\alpha} + \Gamma^{\alpha\nu}_{\sigma}V^{\sigma}) \\ &= \partial^{\mu}\partial^{\nu}V^{\rho} + \partial^{\mu}(\Gamma^{\nu\rho}_{\sigma})V^{\sigma} + \Gamma^{\nu\rho}_{\sigma}(\partial^{\mu}V^{\sigma}) + \Gamma^{\mu\nu}_{\alpha}(\partial^{\alpha}V^{\rho} + \Gamma^{\alpha\rho}_{\sigma}V^{\sigma}) \\ &\quad + \Gamma^{\mu\rho}_{\alpha}(\partial^{\nu}V^{\alpha} + \Gamma^{\alpha\nu}_{\sigma}V^{\sigma}) \end{aligned} \tag{63}$$

and therefore

$$\begin{aligned} &[D^{\mu}, [D^{\nu}, V^{\rho}]] - [D^{\nu}, [D^{\mu}, V^{\rho}]] \\ &\quad = \partial^{\mu}(\Gamma^{\nu\rho}_{\sigma})V^{\sigma} - \partial^{\nu}(\Gamma^{\mu\rho}_{\sigma})V^{\sigma} + \Gamma^{\mu\rho}_{\alpha}\Gamma^{\alpha\nu}_{\sigma}V^{\sigma} - \Gamma^{\nu\rho}_{\alpha}\Gamma^{\alpha\mu}_{\sigma}V^{\sigma} \\ &\quad\quad + \Gamma^{\nu\mu}_{\alpha}(\partial^{\alpha}V^{\rho} + \Gamma^{\alpha\rho}_{\sigma}V^{\sigma}) - \Gamma^{\mu\nu}_{\alpha}(\partial^{\alpha}V^{\rho} + \Gamma^{\alpha\rho}_{\sigma}V^{\sigma}) \\ &\quad = R^{\mu\nu\rho}_{\sigma}V^{\sigma} + \Omega^{\mu\nu}_{\alpha}D^{\alpha}V^{\rho} \end{aligned} \tag{64}$$

where we have introduced the torsion tensor $\Omega_{\alpha}^{\mu\nu} = \Gamma_{\alpha}^{\nu\mu} - \Gamma_{\alpha}^{\mu\nu} = 0$ and the curvature tensor $R_{\sigma}^{\mu\nu\rho}$. Due to the symmetry property of the Christoffel symbols, the curvature tensor is reduced to

$$R_{\sigma}^{\mu\nu\rho}V^{\sigma} = \partial^{\mu}(\Gamma_{\sigma}^{\nu\rho})V^{\sigma} - \partial^{\nu}(\Gamma_{\sigma}^{\mu\rho})V^{\sigma} + \Gamma_{\alpha}^{\mu\rho}\Gamma_{\sigma}^{\alpha\nu}V^{\sigma} - \Gamma_{\alpha}^{\nu\rho}\Gamma_{\sigma}^{\alpha\mu}V^{\sigma} \quad (65)$$

The Jacobi identity gives

$$[D^{\mu}, [D^{\nu}, V^{\rho}]] + [D^{\nu}, [V^{\rho}, D^{\mu}]] + [V^{\rho}, [D^{\mu}, D^{\nu}]] = 0 \quad (66)$$

that is

$$[D^{\mu}, [D^{\nu}, V^{\rho}]] - [D^{\nu}, [D^{\mu}, V^{\rho}]] = [[D^{\mu}, D^{\nu}], V^{\rho}] = 0 \quad (67)$$

and finally

$$[[D^{\mu}, D^{\nu}], V^{\rho}] = R_{\sigma}^{\mu\nu\rho}V^{\sigma} \quad (68)$$

Remark 2. We can also define the Ricci and the electromagnetic energy-momentum tensors, but we were unable to deduce the Einstein equation from this formalism. Naturally, we can write this equation with our brackets as a constraint equation.

Remark 3. We can generalize the covariant derivative by including the skew-symmetric tensor F_{ν}^{μ} in the definition. For this we take into account the gauge curvature for the determination of the new covariant derivative.

For a vectorial function of the velocity we write

$$[\Delta_{\nu}, f^{\mu}(\dot{x})] = \frac{\partial f^{\mu}(\dot{x})}{\partial x^{\nu}} + \Gamma_{\nu\rho}^{\mu} f^{\rho}(\dot{x}) - \frac{q}{m} F_{\rho\nu} \frac{\partial f^{\mu}(\dot{x})}{\partial \dot{x}_{\rho}} \quad (69)$$

and then for the velocity

$$[\Delta_{\nu}, \dot{x}^{\mu}] = \frac{\partial \dot{x}^{\mu}}{\partial x^{\nu}} + \Gamma_{\nu\rho}^{\mu} \dot{x}^{\rho} - \frac{q}{m} F_{\nu}^{\mu} = 0 \quad (70)$$

The covariant derivatives, are then simultaneously covariant under both local internal and external gauges. If we want to keep a synthetic form for the formulas using the curvature and torsion tensors, we must suppose for an arbitrary vector the relation

$$[\Delta_{\nu}, V^{\mu}] = \frac{\partial V^{\mu}}{\partial x^{\nu}} + \Gamma_{\nu\rho}^{\mu} V^{\rho} - A_{\nu} V^{\mu} \quad (71)$$

where the vector A_{ν} is defined by the following equation:

$$F^{\mu\nu} = m\,([\dot{x}^{\mu}, A^{\nu}] - [\dot{x}^{\nu}, A]^{\mu}) \quad (72)$$

Therefore we have

$$[\Delta^\mu, [\Delta^\nu, V^\rho]] - [\Delta^\nu, [\Delta^\mu, V^\rho]] = [[\Delta^\mu, \Delta^\nu], V^\rho]$$
$$= R_\sigma^{\mu\nu\rho} V^\sigma + \Omega_\alpha^{\mu\nu} \Delta^\alpha V^\rho + F^{\mu\nu} V^\rho \quad (73)$$

We define a new “generalized” curvature tensor which matches the local electromagnetism internal symmetry with the local external symmetry:

$$\overline{R}_\sigma^{\mu\nu\rho} V^\sigma = R_\sigma^{\mu\nu\rho} V^\sigma + F^{\mu\nu} V^\rho \quad (74)$$

Then

$$[[\Delta^\mu, \Delta^\nu], V^\rho] = \overline{R}_\sigma^{\mu\nu\rho} V^\sigma \quad (75)$$

6. CONCLUSION

The goal of this work was to study the dynamics associated with the Lorentz-covariant Hamiltonian well known in analytic mechanics. For this, we introduced a four-dimensional bracket structure which gives an algebraic structure, between the position and velocity and generalizes the Poisson brackets. This leads us to introduce a new time parameter which is not the proper time, but is the conjugate coordinate of this covariant Hamiltonian. This formal construction allows us to recover the two groups of Maxwell equations in flat space. This approach is close to the one used by Feynman in his own derivation of the first group of Maxwell equations.

The principal interest of this method, besides the phase space formalism, is in the study of theories with gauges symmetries because it avoids the introduction of the non-gauge-invariant momentum.

Our formalism can be directly extrapolated to curved space, where the principal notions are introduced in a natural manner. A five-dimensional structure can also be studied by considering the τ parameter as a fifth coordinate. In such a case equations take a simpler form, particularly the group of Maxwell equations, but the meaning of this new coordinate is still difficult to interpret, and could be perhaps understood in the context of Kaluza–Klein compactification.

Just after finishing this work we received a paper referring to the covariant Hamiltonian in the context of Feynman’s proof of the Maxwell equations.[11]

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